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ON THE RATIONALITY OF DIVISORS AND MEROMORPHIC FUNCTIONS(1)

BY

CHIA-CHI TUNG

ABSTRACT. Let E be a holomorphic vector bundle over a connected complex manifold X and D a divisor on E. Let $A(D)$ be the set of all $x \in X$ for which $(\operatorname{supp} D) \cap E_x$ is a proper algebraic set in E_x. The purpose of this paper is to prove that the following conditions are equivalent: (i) $A(D)$ has positive measure in X; (ii) D extends to a unique divisor on the projective completion $\bar{E}$ of E; (iii) D is locally given by the divisors of rational meromorphic functions defined over open sets in X. Similar results for meromorphic functions are derived. The proof requires an extension theorem for analytic set: Assume E is a holomorphic vector bundle over a pure p-dimensional complex space X and S an analytic set in E of pure codimension 1. Then the closure $\bar{S}$ of S in $\bar{E}$ is analytic if and only if $S \cap E_x$ is a proper algebraic set for all x in a set of positive $2p$-measure in every branch of X.

1. Statement of results. Let $\pi: E \to X$ be a holomorphic $\mathbf{C}^q$-bundle over a complex space X of dimension p. A meromorphic function m on E is said to be rational (over X) iff on each local trivialization of E, $E(U) \xrightarrow{\sim} U \times \mathbf{C}^q$, m is given by a quotient of (holomorphic) pseudopolynomials defined over U. If S is an analytic set in E, let $A(S)$ denote the set of all $x \in X$ such that $S \cap E_x$ is a proper algebraic set (possibly empty) in $E_x = \pi^{-1}(x)$. If D is a divisor on E, define $A(D) = A(\operatorname{supp} D)$; D is said to be rational over X iff there is an open covering $\{U\}$ of X such that $D|E(U)$ is the divisor of a rational meromorphic function defined on $E(U)$. A subset M of X is said to be of positive s-measure iff it is not of zero Hausdorff s-measure (in terms of local patches of X). The results of this paper are the following.

THEOREM 2.1. *Assume X is nonsingular and irreducible. Let D be a divisor on E. Then the following conditions are equivalent*:

(i) *$A(D)$ has positive $2p$-measure*;

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(ii) *D extends to a unique divisor* $\bar{D}$ *on the projective completion* $\bar{E}$ *of* E *such that* supp $\bar{D} \not\supseteq \bar{E}_\infty$.

(iii) *D is rational over X.*

THEOREM 2.2. *Let X be as above and m a meromorphic function on E. Let* (m) *denote the divisor associated to m.*

(1) *If* $A((m))$ *is of positive 2p-measure, then there exists a rational meromorphic function f on E with* $(f) = (m)$.

(2) *Let* $R(m)$ *be the set of all* $x \in X$ *such that* $m|E_x$ *is a rational meromorphic function. Assume* $R(m)$ *has positive 2p-measure. Then m is rational over X. Furthermore, if X is a Stein manifold with* $H^2(X, \mathbf{Z}) = 0$, *there exist rational holomorphic functions P, Q (which are coprime) on E such that* $m = P/Q$.

The proof of the above theorems requires the following two facts:

THEOREM 1.1. *Assume X is a normal, irreducible complex space of dimension p and g a holomorphic function on E. Assume* $A((g))$ *has positive 2p-measure. Then there exist holomorphic functions* $g_k: E \to \mathbf{C}$ *for* $k = 0, 1, \ldots, r$, *where* g_k *is homogeneous of degree k, and an invertible holomorphic function* $u: E \to \mathbf{C}$ *such that* $\sum_{k=0}^{r} g_k = gu$.

If E is the trivial bundle $\mathbf{C}^p \times \mathbf{C}^q$ over $\mathbf{C}^p$, Theorem 1.1 has been obtained under different hypothesis by L. Ronkin [5] (where $A((g))$ is assumed to be a set of positive Γ-capacity in $\mathbf{C}^p$). In the present case, an elementary construction of a rational representation of (g) is given, making use of the extension of the zero set $g^{-1}(0)$ into the projective completion $\bar{E}$. The extension depends on a generalization of the Remmert-Stein theorem by Stoll [6]. For entire analytic sets, Stoll's theorem is also shown to be valid when the fiber dimension q is greater than 1;

THEOREM 1.2. *Assume X is a pure p-dimensional complex space and* $E \to X$ *a holomorphic* $\mathbf{C}^q$*-bundle with* $q \geq 1$. *Let S be an analytic set in E of pure codimension* 1. *Then the closure* $\bar{S}$ *in* $\bar{E}$ *is analytic if and only if* $A(S)$ *has positive 2p-measure in every branch of X. Moreover, the analyticity of* $\bar{S}$ *implies* codim $\bar{S}_\infty \geq 2$.

It remains unknown if the (sufficiency part of) Theorem 1.2 holds in the case where S has higher codimension.

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2. The holomorphic case. We begin with a lemma without proof.

LEMMA 1. *Let X and Y be complex spaces,* $f: X \times Y \to \mathbf{C}^n$ *a holomorphic map, and S an analytic set in* $X \times Y$. *Assume Y is irreducible. Define*

$$c(f) = \{x \in X | f | \{x\} \times Y = constant\}$$

and

$$d(S) = \{x \in X | \{x\} \times Y \subseteq S\}.$$

Then $c(f)$ and $d(S)$ are analytic in X.

If π: $W \to X$ is a holomorphic fiber space over X and f: $W \to \mathbf{C}$, define $f_x = f|\pi^{-1}(x)$ for each $x \in X$ and $d(f) = \{x \in X | f_x \equiv 0\}$. Also denote the zero-multiplicity of a holomorphic function f at w by $\nu^0(f; w)$.

The proof of Theorem 1.1 will be given in two steps. In the first step we reduce the general case to the case of *a* $\mathbf{C}^1$-bundle. For this purpose, (following an idea of Ronkin [5]) define h: $E \times \mathbf{C} \to \mathbf{C}$ by

$$h(w, t) = g(tw) \qquad ((w, t) \in E \times \mathbf{C}).$$

Let $D = (g)$. By Federer [2, Theorem 2.10.45], the set $\tilde{A}(D) = \pi^{-1}(A(D))$ is of positive measure in E. Clearly for each $w \in \tilde{A}(D)$, the function

$$h_w = h|\{w\} \times \mathbf{C}$$

has a finite zero set. Assume the theorem holds for a $\mathbf{C}^1$-bundle, then h admits a representation

$$h(w, t) = \Phi(w, t)K(w, t)$$

where $\Phi(w, t) = \sum_{k=0}^{r} g_k(w)t^k$ with coefficients g_k holomorphic on E ($g_r \not\equiv 0$), and K is invertible holomorphic on $E \times \mathbf{C}$. Furthermore, K can be chosen so that $K(w, 0) \equiv 1$ for all $w \in E$.(2) Let $E_1 = E - g_r^{-1}(0)$. Define Ψ: $E_1 \times \mathbf{C} \to \mathbf{C}$ by

$$\Psi(w, t) = \prod_{z \in \mathbf{C}} (t - z)^{\nu^0(h_w; (w, z))}.$$

For each $(c,w) \in \mathbf{C}^* \times E_1$, $\Psi(cw, t) = \Psi(w, ct)/c^r$. Let $G = g_r K$. Then $\Psi G = h$ on $E_1 \times \mathbf{C}$, and $\Psi(w, t)G(w, 0) = \Phi(w, t)$. If $(c, w) \in \mathbf{C}^* \times E$ with $h(cw, t) \neq 0$, then

$$G(cw, t) = c^r G(w, ct).$$

Hence

$$\Phi(cw, t) = \Phi(w, ct) \qquad ((c, w, t) \in \mathbf{C} \times E_1 \times \mathbf{C}).$$

It follows that each g_k is homogeneous of degree k along fibers of E, which proves the general case.

Lemma 2. *Assume X and Y are normal, irreducible complex spaces and f, g are holomorphic functions on $X \times Y$. Let π: $X \times Y \to X$ be the projection and define $f_x(g_x)$ with respect to π. Assume $d(f) = \emptyset$. Suppose there exists a dense*

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subset X_0 *of* X_{reg} *such that for all* $x \in X_0$ *and* $w \in \{x\} \times Y_{reg}$,

$$\nu^0(f_x; w) \leqslant \nu^0(g_x; w).$$

Then g/f *is holomorphic on* $X \times Y$.

PROOF. It suffices to prove the lemma for the case where both X and Y are nonsingular. Let $Z = f^{-1}(0) \neq \emptyset$. The projection $\pi_0 = \pi$: $Z \to X$ is open. Let B be a connectivity component of Z_{reg} and $b \in B$. Suppose $\nu^0(f; b) > \nu^0(g; b)$. Then there is an open neighborhood U of b and a holomorphic function h on U such that $f|U = gh$ and $h|B \cap U \equiv 0$. This implies $\pi(B \cap U) \cap X_0 = \emptyset$, a contradiction. Therefore $\nu^0(f; w) \leqslant \nu^0(g; w)$ on Z_{reg}, from which the lemma follows.

LEMMA 3. *Assume* X *is a connected Stein manifold with* $H^2(X, \mathbf{Z}) = 0$. *Let* π: $W \to X$ *be a locally trivial holomorphic fiber space with smooth connected fibers. Let* g *be a holomorphic function* $\not\equiv 0$ *on* W. *Then there exist holomorphic functions* g': $W \to \mathbf{C}$ *and* φ: $X \to \mathbf{C}$ *such that* $g = (\pi^*\varphi) g'$ *and* codim $d(g') \geqslant 2$.

PROOF. Define $Z = g^{-1}(0)$. Let N be the union of all branches of $d(g)$ of codimension 1 in X. If B is a branch of N, then $\pi^{-1}(B)$ is analytic and irreducible in Z, hence by [**1**, Lemma 1.27], $\pi^{-1}(B)$ is a branch of Z. For each $x \in N_{\text{reg}}$ and $w \in Z_{\text{reg}}$ with $\pi(w) = x$, define $\delta(x) = \nu^0(g; w)$. If $x \in X - N$, define $\delta(x) = 0$. Since $\nu^0(g; \cdot)$ is constant on each component of Z_{reg}, the function δ is well defined on $X - N_{\text{sing}}$ and locally constant on N_{reg}. Hence there is a unique divisor D on X whose multiplicity function coincides with δ on $X - N_{\text{sing}}$. Because $H^1(X, \mathcal{O}_X^*) = 0$, there exists a holomorphic function φ: $X \to \mathbf{C}$ with $(\varphi) = D$. Let $V = Z_{\text{sing}} \cup \pi^{-1}(N_{\text{sing}})$. Then V is analytic in W of codimension $\geqslant 2$. For each $w \in W - V$,

$$\nu^0(g; w) \geqslant \nu^0(\pi^*\varphi; w).$$

Hence $g/\pi^*\varphi$ extends to a holomorphic function g': $W \to \mathbf{C}$. Obviously $g = (\pi^*\varphi) g'$. Since g and $\pi^*\varphi$ have the same vanishing order at every point of $Z_{\text{reg}} \cap \pi^{-1}(N_{\text{reg}})$, codim $d(g') \geqslant 2$. Q.E.D.

Let π: $E \to X$ be a holomorphic vector bundle over a complex space X. The associated projective bundle $\mathbf{P}(E)$ is a holomorphic fiber bundle whose fiber over $x \in X$ is the projective space of lines in E_x. Define $\mathbf{C}_X = X \times \mathbf{C}$, $\tilde{E} = \mathbf{C}_X \oplus E$ and $\bar{E} = \mathbf{P}(\tilde{E})$. Then there exist biholomorphic imbeddings of E, resp. $\mathbf{P}(E)$, into $\bar{E}$, such that $\bar{E} = E \cup \mathbf{P}(E)$ and $E \cap \mathbf{P}(E) = \emptyset$. The fiber bundle $\bar{\pi}$: $\bar{E} \to X$ is called the projective completion of E. For each subset V of $\bar{E}$, define $V_\infty = V \cap \mathbf{P}(E)$.

PROOF OF THEOREM 1.1 FOR THE CASE $q = 1$. Let $S = g^{-1}(0)$. If $M \subseteq E$ and $a \in X - d(S)$, define

$$n_g(M_a; 0) = \sum \{\nu^0(g_a; w) | w \in M \cap E_a\}.$$

If M is a relatively compact open set with $\partial M \cap g_a^{-1}(0) = \varnothing$, then $n_g(M_x; 0)$ is constant for all x in a neighborhood of a [7, Theorem 2.7].

If $A(S)$ has interior points, the Remmert-Stein extension theorem implies the closure $\bar{S}$ of S in $\bar{E}$ is analytic of pure codimension 1.[3] In case $A(S)$ is known to have positive measure in X, a generalization of the extension theorem by Stoll [6, Satz 2] yields the analyticity of $\bar{S}$. Consequently $X - d(S) = A(S)$. Define $\Delta(g) = \bar{\pi}(\bar{S}_\infty)$. Since $\operatorname{codim}(\bar{S}_\infty) \geq 2$ (see Theorem 1.2), the analytic set $\Delta(g)$ is nowhere dense in X. It follows that $n_g(E_x; 0) = \text{constant} = r \geq 0$ for all $x \in X - \Delta(g)$.

Let $U \subseteq A(S)$ be an open set over which E is trivial and identify $E(U)$ with $U \times \mathbf{C}$. Let $V = U - \Delta(g)$. Assume $r > 0$. The function

$$G_V(x, t) = \prod_{z \in \mathbf{C}} (t - z)^{\nu^0(g_x; (x,z))} \qquad ((x, t) \in V \times \mathbf{C})$$

is holomorphic on $V \times \mathbf{C}$. For every $x \in V$, let $U_{(x)} = V$ and $G_{(x)} = G_V$. Assume $a \in U \cap \Delta(g)$. There exist open neighborhoods Q of a in U and $Y = \mathbf{C}(\rho)$ $(\rho > 0)$ of the origin in $\mathbf{C}$ such that

$$h(x, t) = \prod_{z \in Y} (t - z)^{\nu^0(g_x; (x,z))}$$

is holomorphic on $Q \times \mathbf{C}$.[4] Then $\tilde{g} = g/h$ is holomorphic on $Q \times \mathbf{C}$ and nonvanishing on $Q \times Y$. Moreover, the function

$$f(x, t) = \prod_{z \in \mathbf{C}} (t - z)^{\nu^0(\tilde{g}_x; (x,z))}$$
$$= f_0(x) + \cdots + f_{l-1}(x) t^{l-1} + f_l(x) t^l$$

is holomorphic on $(Q \cap V) \times \mathbf{C}$ (where $f_l = 1$). By Riemann's extension theorem, each quotient f_j/f_0 extends to a holomorphic function ζ_j on Q. Furthermore, $Q \cap \Delta(g) = \{x \in Q | \zeta_l(x) = 0\}$. Define $U_{(a)} = Q$ and

$$G_{(a)}(x, t) = h(x, t)(\zeta_0(x) + \zeta_1(x) t + \cdots + \zeta_l(x) t^l)$$

on $U_{(a)} \times \mathbf{C}$. Then $G_{(a)}$ is rational holomorphic over $U_{(a)}$. By Lemma 2, $g/G_{(x)}$ is invertible holomorphic on $U_{(x)} \times \mathbf{C}$ for every $x \in U$. Let $G_{(x)} = 1$ if $r = 0$.

Let Ω be a domain in X_{reg} biholomorphic to an open ball in $\mathbf{C}^p$. There exist holomorphic functions φ: $\Omega \to \mathbf{C}$ and g': $\Omega \times \mathbf{C} \to \mathbf{C}$ with $\operatorname{codim} d(g') \geq 2$ and $\varphi(x) g'(x, t) = g(x, t)$ for $(x, t) \in \Omega \times \mathbf{C}$. For each $x \in \Omega - d(g')$, there is a rational holomorphic function $G'_{(x)}$ for which $g'/G'_{(x)}$ is invertible holomorphic on $U_{(x)} \times \mathbf{C}$. Hence there exists a rational holomorphic function

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([3])Since it can be shown that $\bar{\pi}(\bar{S}_\infty) \cap U = \varnothing$ for some nonvoid open set $U \subseteq X$.

([4])Cf. [8, Theorem 1.2.20] or [7, Theorem 3.7].

G'_Ω: $\Omega \times \mathbf{C} \to \mathbf{C}$ for which g'/G'_Ω is invertible holomorphic. Define $G_\Omega = (\pi^*\varphi)G'_\Omega$ on $E(\Omega)$. Observe that if $x \in \Omega - \Delta(g')$, $n_{g'}(E_x; 0) = r$.

Choose an open covering $\{\Omega_j\}$ of X_{reg} such that $g|E(\Omega_j) = G_j \cdot H_j$ where G_j is rational holomorphic over Ω_j and H_j invertible holomorphic on $E(\Omega_j)$ with $H_j = 1$ on the zero section of $E(\Omega_j)$. Then there exist holomorphic functions g_k on E_{reg} for $k = 0, 1, \dots, r$ such that

$$\left(\sum_{k=0}^{r} g_k\right) = (g)|E_{\text{reg}},$$

and each g_k is homogeneous of degree k along fibers of E. Since X is normal, the above equation extends to E. Q.E.D.

Proof of Theorem 1.2. It is easy to show that the analyticity of $\bar{S}$ implies $X - d(S) = A(S)$ and codim $d(S) \geqslant 1$. Conversely, assume $A(S)$ has positive $2p$-measure in every branch of X. If $q = 1$, the analyticity of $\bar{S}$ follows from Satz 2 of Stoll [6]. Now assume $q > 1$ and X is nonsingular. Let X_ν be a connectivity component of X. There exists an open set $\Omega \subseteq X_\nu$ biholomorphic to an open ball in $\mathbf{C}^p$ such that $A(S) \cap \Omega$ has positive $2p$-measure. Let D be the divisor on $E(\Omega)$ associated to the analytic set $S(\Omega) = S \cap E(\Omega)$. By Theorem 1.1, there exists a holomorphic function $g = \Sigma_{k=0}^{r} g_k$ on $E(\Omega)$, where each g_k is homogeneous of degree k (with $g_r \not\equiv 0$), such that $(g) = D$. The equation

$$\sum_{k=0}^{r} z^{r-k} g_k(w) = 0$$

defines an analytic set V in $\bar{E}(\Omega)$ of pure codimension 1. Moreover, $V \cap E(\Omega) = S(\Omega)$ and hence $\bar{S} \cap \bar{E}(\Omega) = V$. The Remmert-Stein extension theorem implies $\bar{S}$ is analytic in $\bar{E}$ of pure codimension 1.

If X is singular, let $M = \bar{E} - \mathbf{P}(E(X_{\text{sing}}))$ and $N = \bar{S} \cap M$. The analyticity of $\bar{S} \cap \bar{E}(X_{\text{reg}})$ in $\bar{E}(X_{\text{reg}})$ implies that of N in M, and N has pure codimension 1. Since dim $N >$ dim $\mathbf{P}(E(X_{\text{sing}}))$, it follows that $\bar{S} = \bar{N}$ is analytic in $\bar{E}$. Now suppose $\bar{S}_\infty$ contains a branch B of $\bar{S}$. Then the projection map τ: $B \to X$ is proper, holomorphic, of pure rank p. Hence $X' = \tau(B)$ is a branch of X, and for every $x \in X'$, $(\bar{E}_\infty)_x = B \cap \tau^{-1}(x) \subseteq \bar{S}_x$. On the other hand, it can be shown that X' contains an open set Ω for which $\bar{E}(\Omega)_\infty \not\subseteq \bar{S}(\Omega)$. Hence a contradiction. Thus codim $\bar{S}_\infty \geqslant 2$.

3. Application to divisors and meromorphic functions.

Proof of Theorem 2.1. It is easy to show that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Assume D satisfies (i) and let S = supp D. By Theorem 1.2, $\bar{S}$ is analytic in $\bar{E}$ of pure codimension 1. Let U be an open set in X biholomorphic to an open ball in $\mathbf{C}^p$. Then $E(U)$ is a connected Stein manifold with $H^2(E(U), \mathbf{Z}) = 0$. There exist relatively prime holomorphic functions f_U and g_U on $E(U)$ such that

$$(f_U/g_U) = D|E(U).$$

For every $x \in A(D) \cap U$, $(f_U)_x^{-1}(0)$ and $(g_U)_x^{-1}(0)$ are algebraic sets in E_x of pure codimension 1, provided nonempty. Then $d(S)$ is nowhere dense analytic in X and $U - d(S) = A(D) \cap U$. Theorem 1.1 implies there are relatively prime rational holomorphic functions P_U and Q_U on $E(U)$ such that

$$(P_U/Q_U) = D|E(U).$$

Thus the theorem is proved.

Observe also that the polynomial bidegree of a relatively prime rational representation (P_U, Q_U) of D on $E(U)$ is independent of the open set U.

PROOF OF THEOREM 2.2. (1) By Theorem 2.1, there exists an open covering $\{U\}$ of X such that

$$m|E(U) = g_U h_U$$

where g_U is a rational meromorphic function over U and h_U is invertible holomorphic on $E(U)$ with $h_U \equiv 1$ on the zero section of $E(U)$. Consequently there is a rational meromorphic function f on E for which $(f) = m$.

(2) Suppose $m \not\equiv 0$. There exists an open set Ω in X for which (i) $m|E(\Omega)$ is given by a quotient f/g of relatively prime holomorphic functions f, g; (ii) $\Omega \cap R(m)$ has positive $2p$-measure. Let $N = \{x \in \Omega | f_x \cdot g_x \equiv 0\}$. Lemma 1 implies $(\Omega - N) \cap R(m)$ has positive $2p$-measure. Hence there exists a rational meromorphic function f' on E with $(f') = (m)$. Define $u = m/f'$. The analytic set $c(u) = R(u)$. Since $R(u)$ has positive $2p$-measure in X, $c(u) = X$. Therefore there exists a holomorphic function φ on X for which $m = (\pi^*\varphi)f'$. Hence m is rational over X.

Assume X is Stein with $H^2(X, \mathbf{Z}) = 0$. There exist relatively prime holomorphic functions g_j $(j = 1, 2)$ on E such that $m = g_1/g_2$. Each $A((g_j))$ has positive $2p$-measure. Hence there is a rational holomorphic function f_j on E (given by Theorem 1.1) with $(f_j) = (g_j)$. Then $m/(f_1/f_2)$ is invertible holomorphic on E. As above, there exists an invertible holomorphic function Ψ on X for which

$$m = (\pi^*\Psi)\frac{f_1}{f_2}. \quad \text{Q.E.D.}$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

Current address: Department of Mathematics, Columbia University, New York, New York 10027.